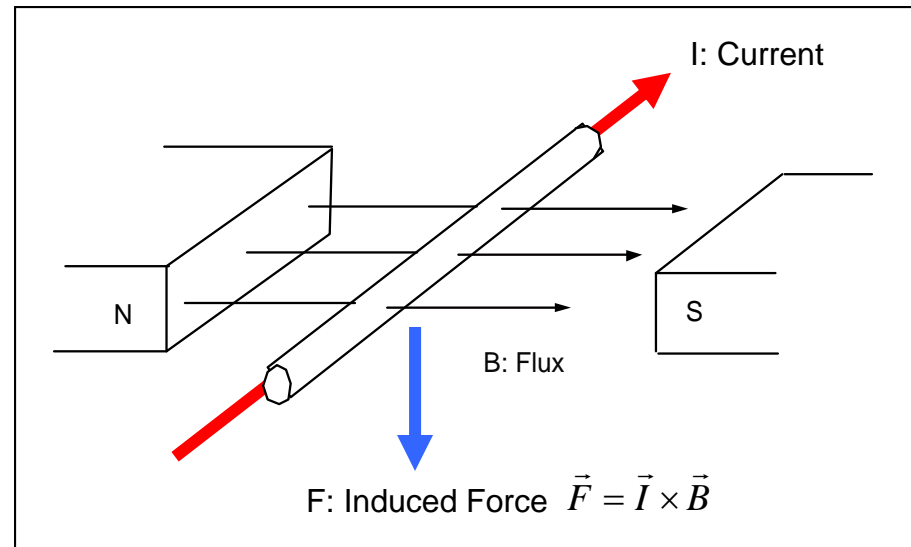


# DC Motor Theory

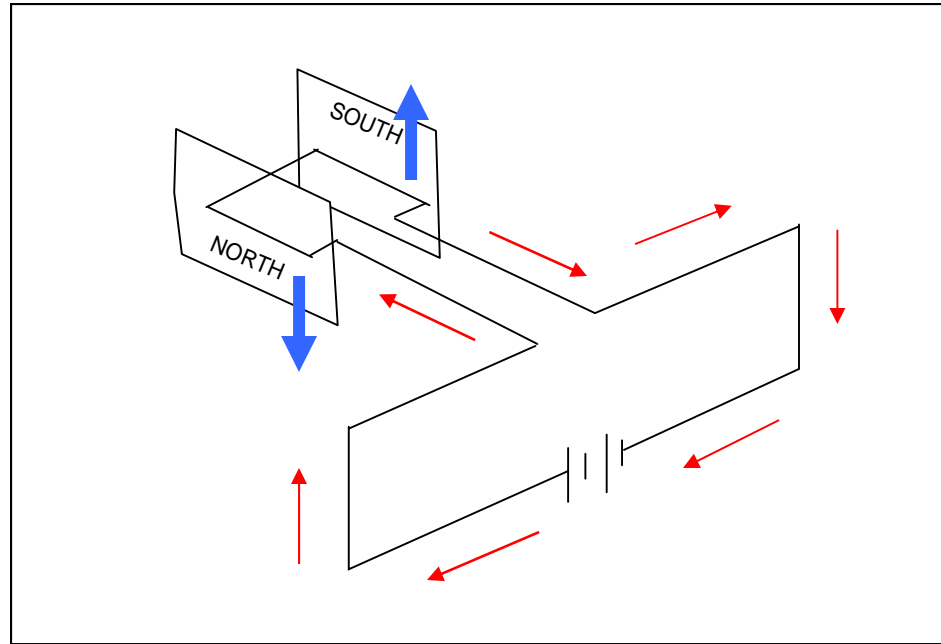
**Objective:** To understand and derive DC motor dynamics

1. Lorentz's Law of electromagnetic forces



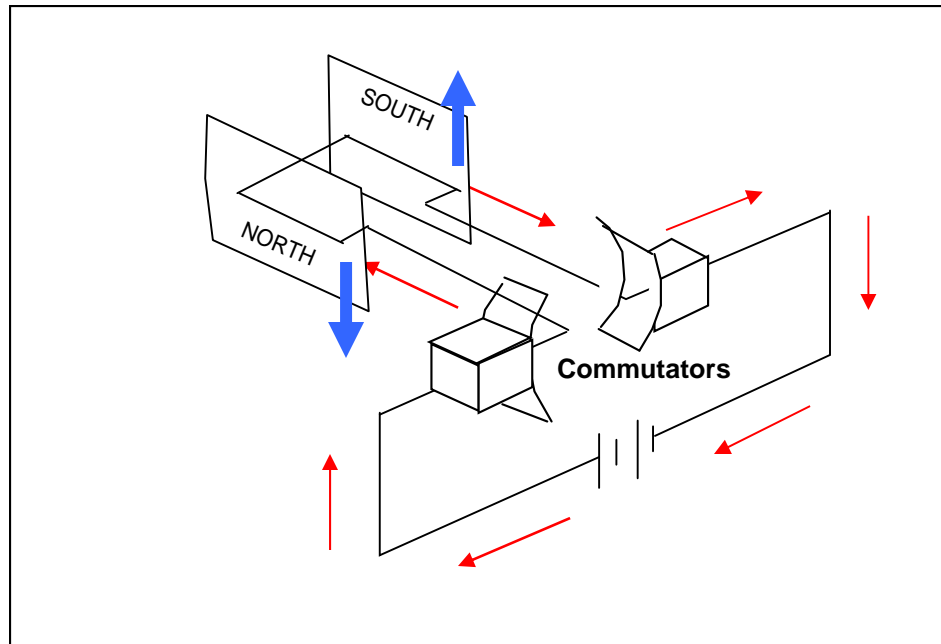
**Lorentz's Law:** *a current-carrying wire in a magnetic field will induce an electromotive force*

## 2. Loop of wire: What happens?



- A. Current flows in direction shown (red arrows)
- B. Left part of loop moves down. Right part moves up (blue arrows)

### 3. Creating rotary motion



- C. Commutators allow loop to rotate 180 degrees
- D. Current then reverses direction
- E. Consequently have constant CCW motion

#### 4. Motor Property 1 – Torque is proportional to current

**Question:** What is an inductor?

**Answer:** Another name for a coil or loop of wire

Inductors oppose the change in current. Mathematically this is:

$$V_L = L \frac{dI}{dt} \quad \left\{ \begin{array}{l} L: \text{ inductance [H]} \\ V_L: \text{ voltage drop [V] across the inductor} \\ I: \text{ current [A]} \end{array} \right. \quad (1)$$

As motor turns, then:

- A.  $dI/dt$  increases and hence  $V_L$  increases
- B. Induced voltage opposes applied voltage and limits current
- C. Induced voltage is called back EMF

$$V = IR + e \quad \left\{ \begin{array}{l} V: \text{ Applied voltage} \\ I: \text{ Current through coil} \\ R: \text{ Coil resistance} \\ e: \text{ Back EMF (induced by coil rotation)} \end{array} \right. \quad (2)$$

$$V = IR + e$$

$$e = K_e \omega \quad \left\{ \begin{array}{l} \text{Back EMF proportional to coil's rotational speed } \omega \\ K_e \text{ is called the Back EMF constant} \end{array} \right. \quad (3)$$

Hence from (2) have:  $V = IR + K_e \omega$  (4)

D. Torque coil makes available is proportional to current:

$$T = K_t I \quad \left\{ K_t \text{ is called the torque constant} \right. \quad (5)$$

NB: torque is independent of voltage

E. **Lemma** – the back EMF and torque constants are equal

**Proof:** Mechanical power output by shaft equals electrical power minus heating losses

$$P_m = P_e - I^2 R \quad \left\{ \begin{array}{l} P_m \text{ is the mechanical power} = T \omega \\ P_e \text{ is the electrical power} = VI \end{array} \right. \quad (6)$$

Hence (6) becomes

$$T \omega = VI - I^2 R \quad (7)$$

Subbing (5) and (4) into (7) yields

$$\begin{aligned}K_t I \omega &= (IR + K_e \omega) I - I^2 R \\ &= I^2 R + K_e \omega I - I^2 R \\ &= K_e I \omega\end{aligned}$$

Therefore proved that

$$K_t = K_e \tag{8}$$

5. Motor Property 2 – Motor speed increases then torque decreases

With Lemma (8) can rewrite (4) as

$$V = \left( \frac{T}{K} \right) R + K \omega \tag{9}$$

Alternatively

$$\omega = -\frac{TR}{K^2} + \frac{V}{K} \tag{10}$$

- Equation (10) says as motor speed increases, the torque decreases

## 6. Motor Dynamics – Equations of Motion

$$\text{Newton:} \quad J\ddot{\theta} = T = KI \quad (11A)$$

$$\text{Lorentz:} \quad V = L\frac{dI}{dt} + IR + K\dot{\theta} \quad (11B)$$

Taking Laplace yields:

$$Js\Omega = KI \quad (12A)$$

$$V = LsI + IR + Ks\Omega \quad (12B)$$

Consequently have

$$I = \frac{Js\Omega}{K}$$

Substituting into (12B) yields

$$V = \frac{Ls^2 J\Omega}{K} + \frac{Js\Omega R}{K} + K\Omega = \Omega \left( \frac{JLs^2}{K} + \frac{JsR}{K} + K \right) = \Omega \left( \frac{s^2 JL + sJR + K^2}{K} \right)$$

Hence

$$\frac{\Omega}{V} = \frac{K}{s^2 JL + sJR + K^2}$$

**Second order ODE.**  
Time plot?



$$\frac{\Omega}{V} = \frac{K}{s^2 JL + sJR + K^2}$$

If inductance is low, then

$$\frac{\Omega}{V} = \frac{K}{sJR + K^2}$$

First order ODE  
Time plot?