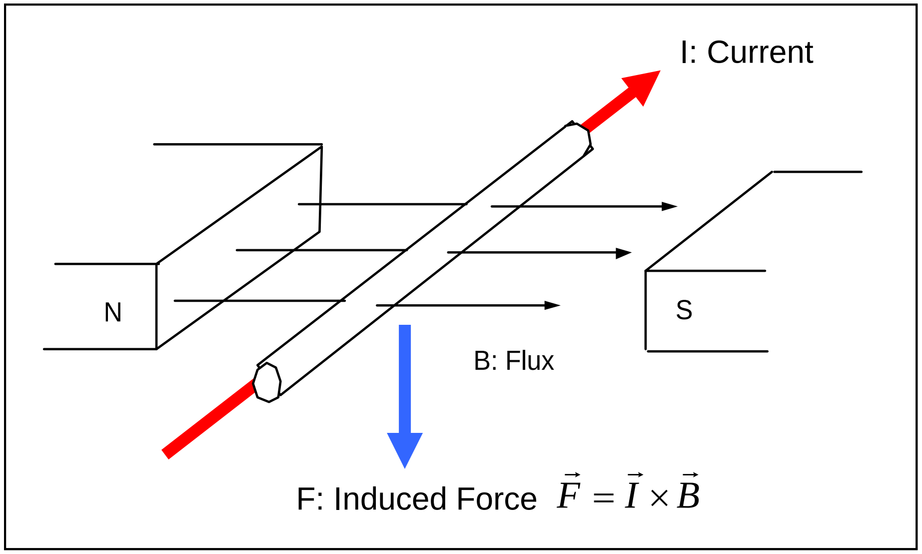
DC Motor Theory

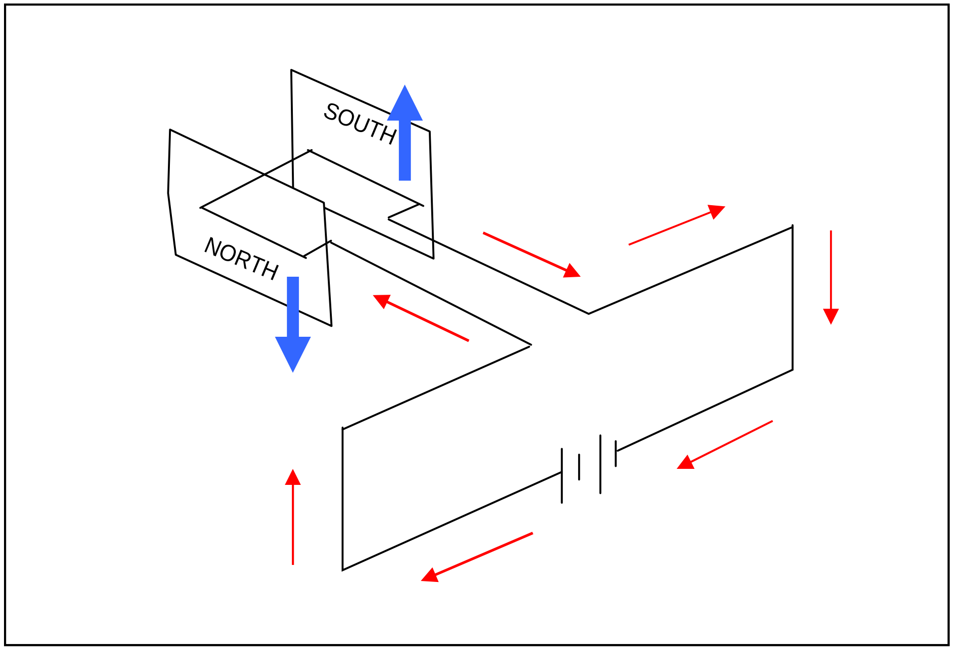
Objective: To understand and derive DC motor dynamics

1. Lorentz's Law of electromagnetic forces



Lorentz's Law: *a current-carrying wire in a magnetic field will induce an electromotive force*

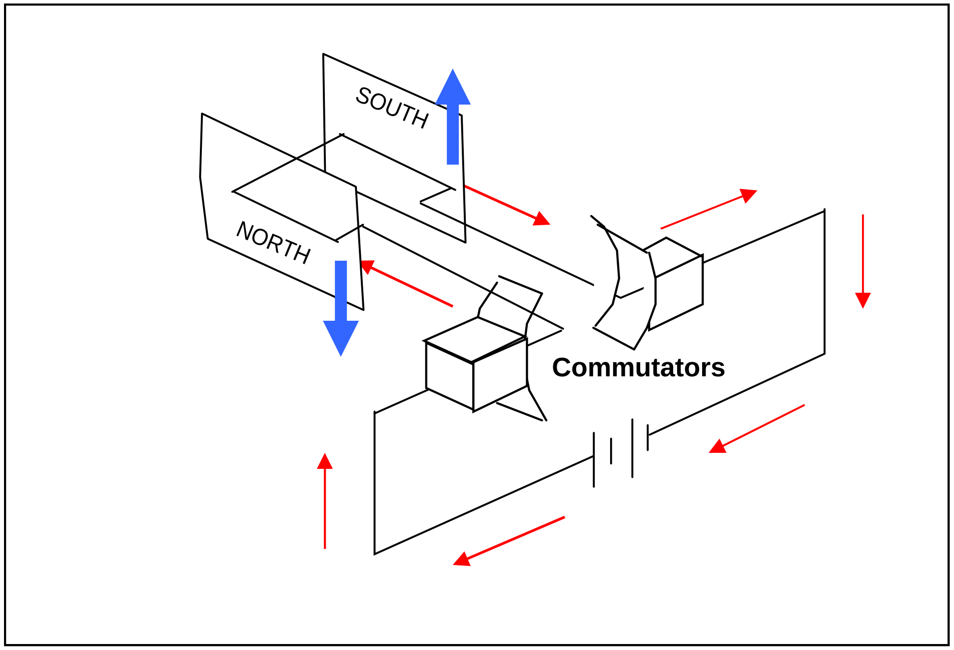
2. Loop of wire: What happens?



A. Current flows in direction shown (red arrows)

B. Left part of loop moves down. Right part moves up (blue arrows)

3. Creating rotary motion



C. Commutators allow loop to rotate 180 degrees

D. Current then reverses direction

E. Consequently have constant CCW motion

4. Motor Property 1 – Torque is proportional to current

Question: What is an inductor?
Answer: Another name for a coil or loop of wire

Inductors oppose the change in current. Mathematically this is:

$$V_L = L\frac{dI}{dt} \qquad \begin{cases} L: & \text{inductance [H]} \\ V_L: & \text{voltage drop [V] across the inductor} \\ I: & \text{current [A} \end{cases} \tag{1}$$

As motor turns, then:

A. dI/dt increases and hence V_L increases

B. Induced voltage opposes applied voltage and limits current

C. Induced voltage is called back EMF

$$V = IR + e \qquad \begin{cases} V: & \text{Applied voltage} \\ I: & \text{Current through coil} \\ R: & \text{Coil resistance} \\ e: & \text{Back EMF (induced by coil rotation} \end{cases} \tag{2}$$

$$V = IR + e$$

$$e = K_e \omega$$

Back EMF proportional to coil's rotational speed ω

K_e is called the Back EMF constant (3)

Hence from (2) have: $$V = IR + K_e \omega \quad (4)$$

D. Torque coil makes available is proportional to current:

$$T = K_t I \quad (5)$$

K_t is called the torque constant

NB: torque is independent of voltage

E. **Lemma** – the back EMF and torque constants are equal

Proof: Mechanical power output by shaft equals electrical power minus heating losses

$$P_m = P_e - I^2 R \quad (6)$$

P_m is the mechanical power = $T\omega$

P_e is the electrical power = VI

Hence (6) becomes

$$T\omega = VI - I^2 R \quad (7)$$

Subbing (5) and (4) into (7) yields

$$\begin{aligned} K_t I\omega &= \left(IR + K_e\omega\right)I - I^2R \\ &= I^2R + K_e\omega I - I^2R \\ &= K_e I\omega \end{aligned}$$

Therefore proved that

$$K_t = K_e \tag{8}$$

5. Motor Property 2 – Motor speed increases then torque decreases

With Lemma (8) can rewrite (4) as

$$V = \left(\frac{T}{K}\right)R + K\omega \tag{9}$$

Alternatively

$$\omega = -\frac{TR}{K^2} + \frac{V}{K} \tag{10}$$

- Equation (10) says as motor speed increases, the torque decreases

6. Motor Dynamics – Equations of Motion

Newton: $$J\ddot{\theta} = T = KI \qquad (11A)$$

Lorentz: $$V = L\frac{dI}{dt} + IR + K\dot{\theta} \qquad (11B)$$

Taking Laplace yields:

$$Js\Omega = KI \qquad (12A)$$

$$V = LsI + IR + Ks\Omega \qquad (12B)$$

Consequently have

$$I = \frac{Js\Omega}{K}$$

Substituting into (12B) yields

$$V = \frac{Ls^2 J\Omega}{K} + \frac{Js\Omega R}{K} + K\Omega \quad = \Omega\left(\frac{JLs^2}{K} + \frac{JsR}{K} + K\right) \quad = \Omega\left(\frac{s^2 JL + sJR + K^2}{K}\right)$$

Hence

$$\frac{\Omega}{V} = \frac{K}{s^2 JL + sJR + K^2}$$

Second order ODE.
Time plot?

$$\frac{\Omega}{V} = \frac{K}{s^2 JL + sJR + K^2}$$

If inductance is low, then

$$\frac{\Omega}{V} = \frac{K}{sJR + K^2}$$

First order ODE

Time plot?