Open-Loop Step Response

Recall that the dynamic equations of motion for a DC motor (for low inductances) can be modeled as a 1st order system.

Modeling can be also be represented by Block Diagrams and Laplace Transforms

$$\underset{\text{motor command}}{V(s)} \longrightarrow \underbrace{\boxed{\frac{b}{s+a}}}_{G_{OL}(s)} \longrightarrow \underset{\text{motor velocity}}{\Omega(s)}$$

$G_{OL}(s)$: called Open Loop Transfer Function

Mathematically, this becomes:

(1)

or

$$(s+a)\Omega(s) = bV(s)$$

Method 1: Solve for $\omega(t)$ with Ordinary Differential Equations:

(1) becomes: (2)

Suppose one gives the motor a step input i.e. applies a velocity command

In time domain, step input is:

(3)

Subbing (3) into (2) yields

$$\dot{\omega}(t) + a\omega(t) = bM \tag{4}$$

This first-order differential equation is solved using an integrating factor

when one has equations of the form:

$$\dot{\omega} + p(t)\omega = g(t) \tag{5}$$

Thus comparing (4) and (5), say:

and using integrating factor, one creates the equation:

$$e^{at}\{\dot{\omega} + a\omega\} = e^{at}bM$$

or

(6)

Integrating (6) yields

$$e^{at}\omega = Mb\int e^{at}\,dt = Mb\left\{\frac{1}{a}e^{at} + C\right\} \quad \text{where C is unknown constant} \qquad (7)$$

Multiplying (7) with the exponent yields:

or

$$\omega(t) = \frac{M}{a}b + De^{-at} \quad \text{where D is unknown constant} \qquad (8)$$

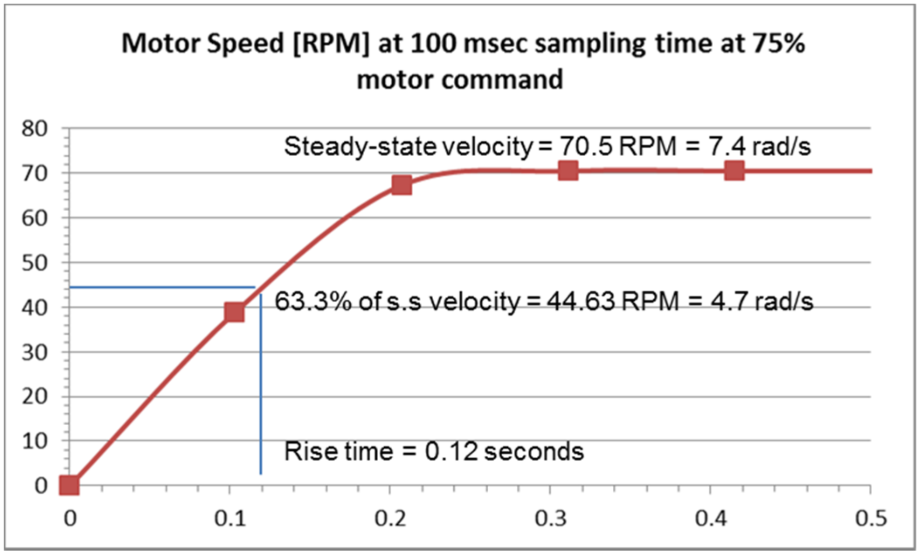
With initial conditions (IC), say that $\omega(0) = 0$ then (8) becomes

or (9)

Subbing (9) into (8), one gets general solution form

$$\omega(t) = \frac{Mb}{a} - \frac{Mb}{a}e^{-at} = \frac{Mb}{a}(1 - e^{-at}) \qquad (10)$$

From M=75 step response lab of NXT DC motor, we saw a graph like:



$$\omega(t) = \omega_{ss}(1 - e^{-at}) \tag{11}$$

If we say that $a \triangleq \frac{1}{\tau}$ then one has

$$\omega(t) = \omega_{ss}\left(1 - e^{-\frac{t}{\tau}}\right)$$

From graph, see $\omega_{ss} = 70.5$ RPM

Brackets	ω	Error [V]
$1-e^{-1}=0.63$	44.4	70.5 – 44.4 = 26.1 or 37% (i.e. **63%** of steady-state)
$1-e^{-2}=0.86$	60.6	70.5 – 60.6 = 9.9 or 14.0% (i.e. 86% of steady-state)
$1-e^{-3}=0.95$	67.0	70.5 – 67.0 = 3.5 or 5% (i.e. **95%** of steady-state)
$1-e^{-2}=0.86$	69.1	70.5 – 69.1 = 1.4 or 2% (i.e. 98% of steady-state)
$1-e^{-1}=0.63$	69.8	70.5 – 69.8 = 0.7 or 1% (i.e. **99%** of steady-state)

Comparing (10) and (11) we see that:

(12)

Subbing (11) and (12) into (1) yields:

$$G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} = \frac{\omega_{ss}}{M\tau}\frac{1}{s+\frac{1}{\tau}} \tag{13}$$

For $\tau = 0.12$ and $\omega_{ss} = 70.5$ RPM, and step input $M = 75$ (13) becomes:

$$\boxed{G_{OL}(s) = \frac{70.5}{75(0.12)}\frac{1}{s+\frac{1}{0.12}} = \frac{7.83}{s+8.33}} \tag{14}$$

Method 2: Solve for $\omega(t)$ with Laplace Transforms:

Like before, and referring to (1), have

$$V(s) \longrightarrow \boxed{\frac{b}{s+a}} \longrightarrow \Omega(s) \qquad G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} \qquad \text{from (1)}$$

Like before, one has a step input i.e. velocity command, referring to (3)

$$v(t) = \begin{cases} 0: t \leq 0 \\ M: t > 0 \end{cases} \qquad \text{from (3)}$$

The Laplace Transform for a step input signal is given by:

(15)

Substituting (15) into (1) yields:

(16)

The inverse Laplace of (16) yields response in time-domain:

$$\omega(t) = \frac{Mb}{a}\left(1 - e^{-at}\right) \tag{17}$$

If we define:

(18)

and from NXT motor plot that steady-state, have

(19)

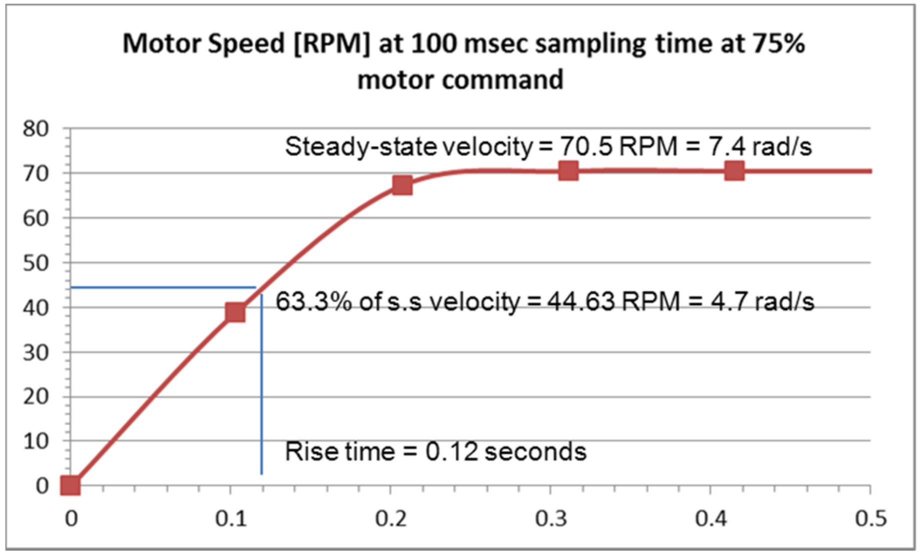
Then with (18) and (19), (17) yields:

$\omega_{ss} = Mb\tau$ or (17)

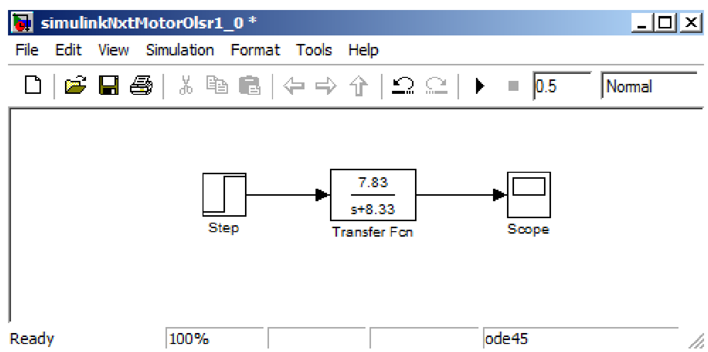
Hence, subbing values of a and b in (1), yields:

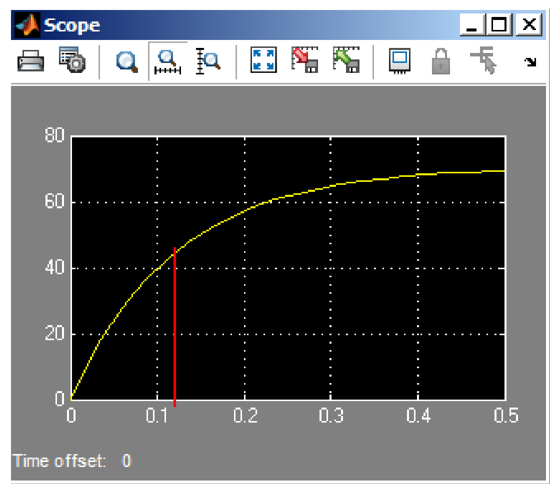
$$\boxed{G_{OL}(s) = \frac{\Omega(s)}{V(s)} = \frac{b}{s+a} = \frac{\omega_{ss}}{M\tau}\frac{1}{s+\frac{1}{\tau}} = \frac{70.5}{75(0.12)}\frac{1}{s+\frac{1}{0.12}} = \frac{7.83}{s+8.33}} \tag{18}$$

Same as (14); Laplace gives same
Solution as ODE method



Simulink simulation of OLTF yields a plot similar to experimentally acquired one:





In Lab, will compare experimental and (Simulink) simulated OLTF step response