ME729 Advanced Robotics - Homogeneous Transformations

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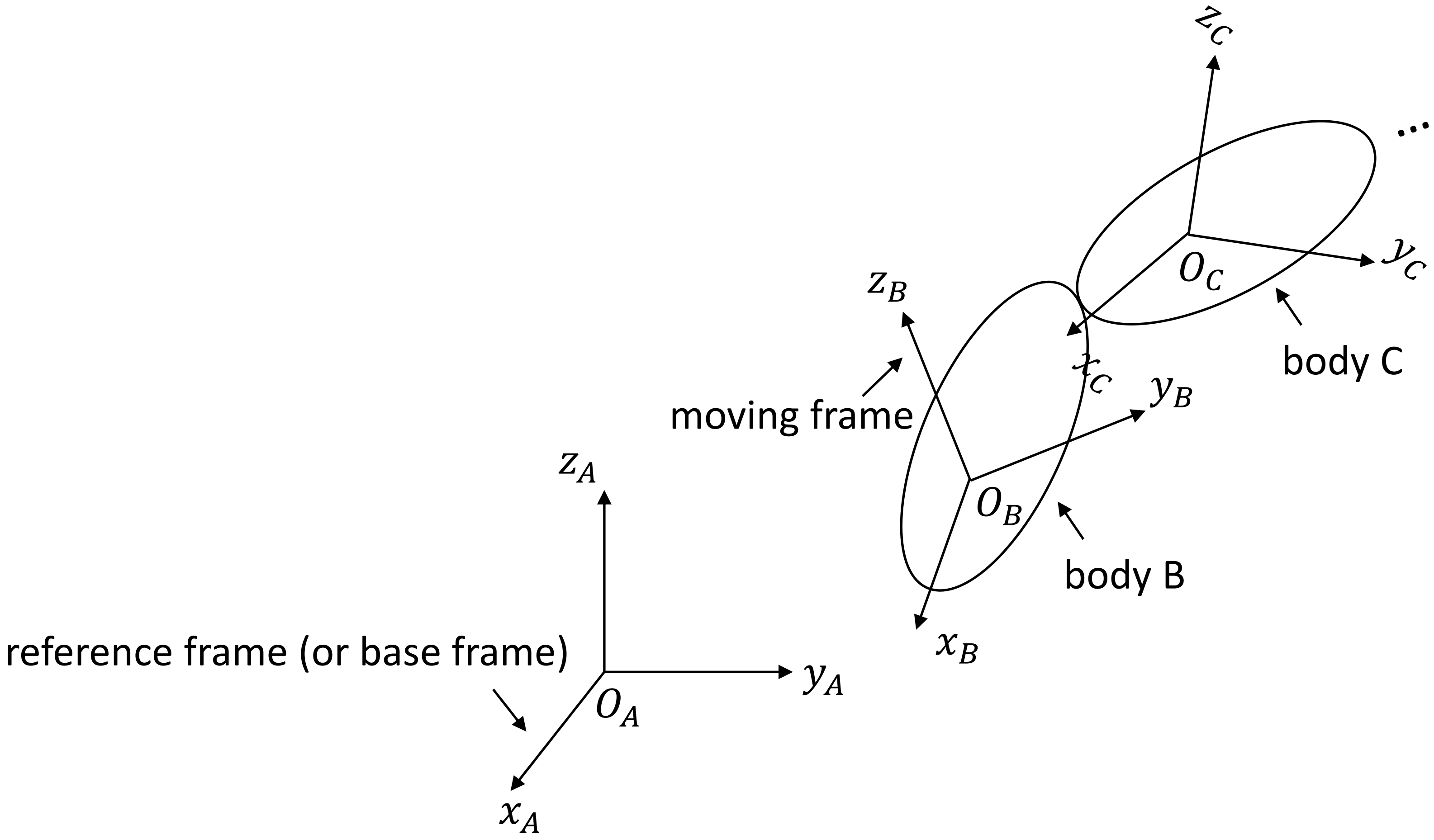
Coordinate Frames



- A reference or base frame is a fixed frame on ground.
- Moving (or body-fixed) frames are fixed frames on each body.

Homogeneous Coordinates

- Homogeneous coordinates is that an *n*-space vector is represented as an (*n*+1)-space vector with a scale factor.
- Cartesian space (*n* = 3) case,

$$\boxed{v = a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}} \longrightarrow \boxed{\begin{gathered} v = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}, \text{where } w \text{ is a scale factor and } w \neq 0 \\ a = \frac{x}{w}, b = \frac{y}{w}, c = \frac{z}{w} \end{gathered}}$$

- For example,

$$v = 2\boldsymbol{i} + 5\boldsymbol{j} + 6\boldsymbol{k} \longrightarrow v = \begin{bmatrix} 2 \\ 5 \\ 6 \\ 1 \end{bmatrix} \; or \; \begin{bmatrix} 4 \\ 10 \\ 12 \\ 2 \end{bmatrix}, \text{etc}$$

- In robotics, the scale factor, w, is one ($w = 1$).

Translation Transformation

- The transformation H corresponding to a translation by a vector $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$ is

$$\mathrm{H} = \mathrm{Trans}(a, b, c) = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Given a vector $\mathrm{u} = [x, y, z, w]^T$, the transformed vector **v** is given by

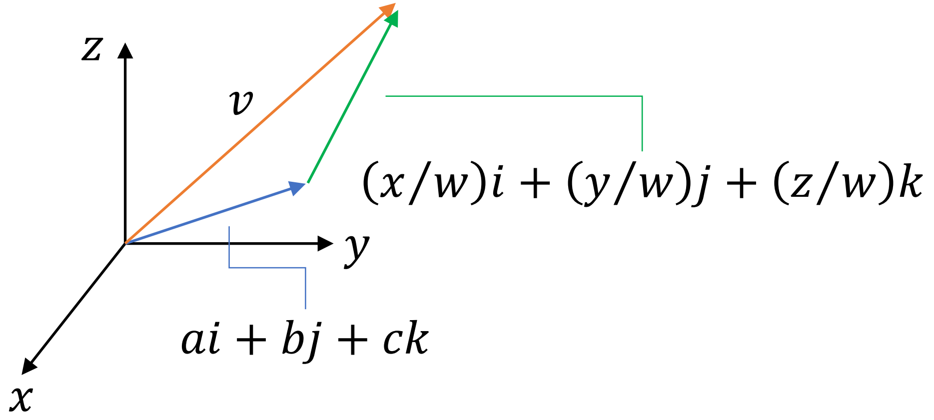
$$v = \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x + aw \\ y + bw \\ z + cw \\ w \end{bmatrix} = \begin{bmatrix} x/w + a \\ y/w + b \\ z/w + c \\ 1 \end{bmatrix}$$

- For example,

$$\begin{bmatrix} 6 \\ 0 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

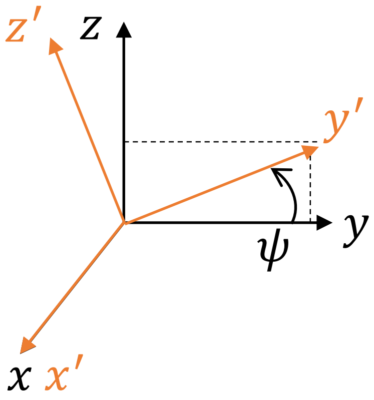
* $2i + 3j + 2k$ translated by, or added to $4i - 3j + 7k$

* The translation is interpreted as the addition of the two vectors, $(x/w)i + (y/w)j + (z/w)k$ and $ai + bj + ck$



Rotation Transformations

- A rotation about *x* axis



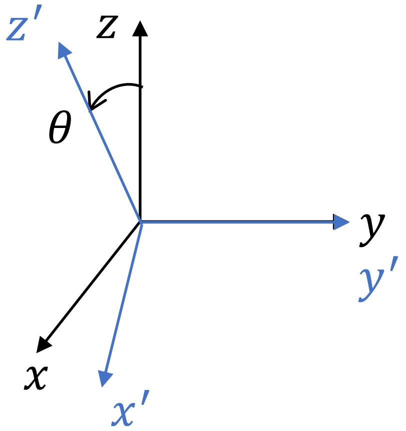
$$x' = x$$

$$y' = \cos(\psi)\, j + \sin(\psi)\, k$$

$$z' = -\sin(\psi)\, j + \cos(\psi)\, k$$

$$\mathrm{Rot}(x,\psi) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\psi) & -\sin(\psi) & 0 \\ 0 & \sin(\psi) & \cos(\psi) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A rotation about *y* axis



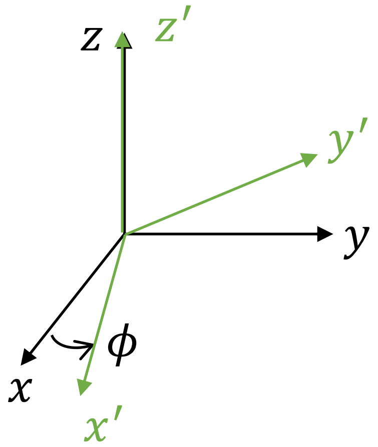
$$x' = \cos(\theta)\, i - \sin(\theta)\, k$$

$$y' = y$$

$$z' = \sin(\theta)\, i + \cos(\theta)\, k$$

$$\mathrm{Rot}(y,\theta) = \begin{bmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- A rotation about *z* axis



$$x' = \cos(\phi)\, i + \sin(\phi)\, j$$

$$y' = -\sin(\phi)\, i + \cos(\phi)\, j$$

$$z' = z$$

$$\mathrm{Rot}(z,\phi) = \begin{bmatrix} \cos(\phi) & -\sin(\phi) & 0 & 0 \\ \sin(\phi) & \cos(\phi) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Rotation Transformations

- Given $u = 7i + 3j + 2k$, rotate it about *z* axis with $90°$.

$$\therefore \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ 2 \\ 1 \end{bmatrix}$$

- Given $u = 7i + 3j + 2k$, rotate it about *z* axis with $90°$ and about *y* axis with $90°$, and translate it along $4i - 3j + 7k$.

$$\text{Trans(4, -3, 7)Rot(y, 90)Rot(z, 90)} = \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \\ 10 \\ 1 \end{bmatrix}$$

Homogeneous Transformations

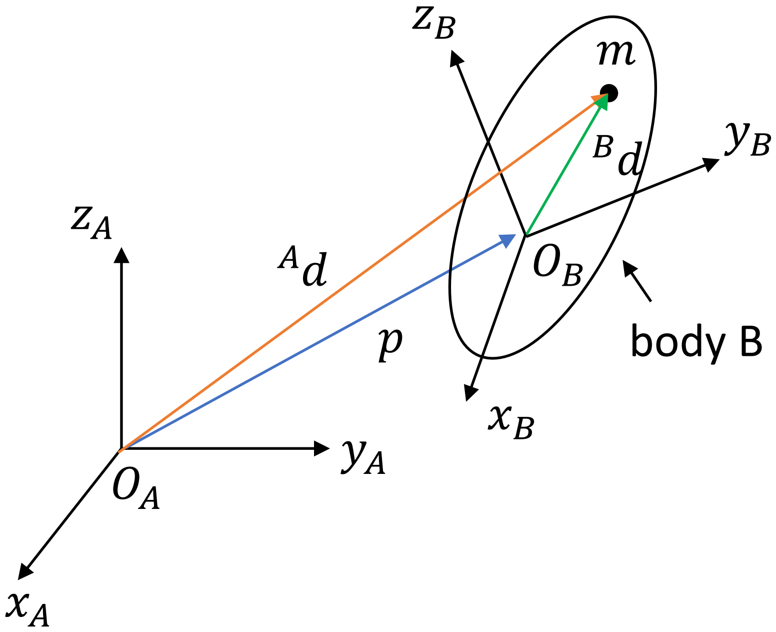
- The general representation of homogeneous transformation matrix

$$T = \begin{array}{c} \begin{array}{cccc} \boldsymbol{n} & \boldsymbol{o} & \boldsymbol{a} & \end{array} \\ \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Translation transformation $(\boldsymbol{p})$

Rotation transformation matrix $(\boldsymbol{R})$

$$= \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}$$



- The frame A is a reference or base frame and the frame B is a body-fixed frame.
- There is a *m* point on body C.
- Usually, the representation of $^B d$ w.r.t the frame B is easy.
- But we need to transform $^B d$ into $^A d$ w.r.t the base frame.

- $^A d = {}^A T_B {}^B d = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} {}^B d$

Inverse Homogeneous Transformations

- The inverse of the rotation matrix R is equal to its transpose.

$$\boxed{R^{-1} = R^T}$$

- The inverse of the homogeneous transformation matrix can be obtained as

$$T^{-1} = \begin{bmatrix} \boldsymbol{R}^T & -\boldsymbol{R}^T\boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} = \begin{bmatrix} n_x & n_y & n_z & -\boldsymbol{n}\cdot\boldsymbol{p} \\ o_x & o_y & o_z & -\boldsymbol{o}\cdot\boldsymbol{p} \\ a_x & a_y & a_z & -\boldsymbol{a}\cdot\boldsymbol{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \left(\text{* Remind } T = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix}\right)$$

$\boldsymbol{R}^T$

- The inverse of a matrix ${}^{A}T_{B}$ is the matrix ${}^{B}T_{A}$.

$$\boxed{{}^{B}T_{A} = {}^{A}T_{B}^{-1}}$$

  - ${}^{A}T_{B}$ is that consider a frame B w.r.t a frame A.
  - ${}^{B}T_{A}$ is that consider a frame A w.r.t a frame B.