ME729 Advanced Robotics - Forward Kinematics

2/5/2018

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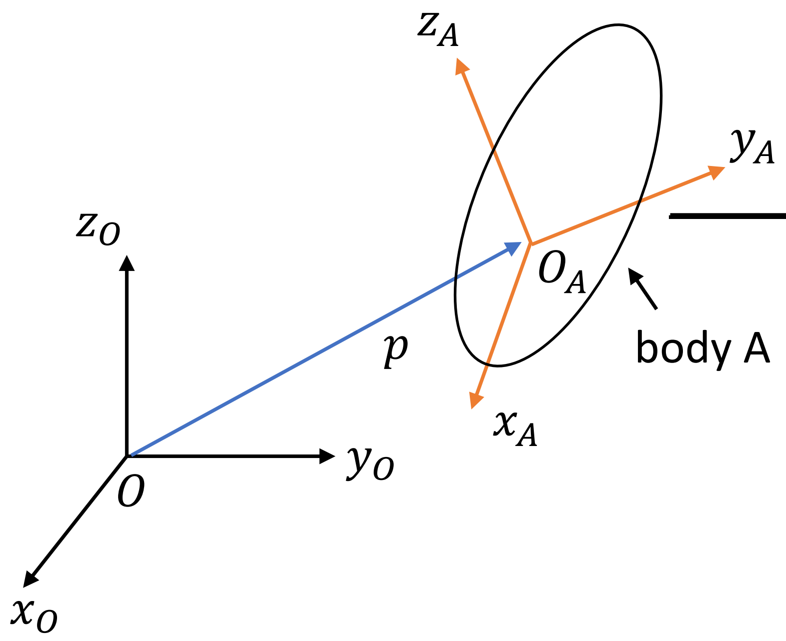
Representation of Orientation

- Three common methods to represent orientation
    - There are a base frame and a moving frame A.
    - We want to know how many rotations is the frame A made w.r.t. the base frame.



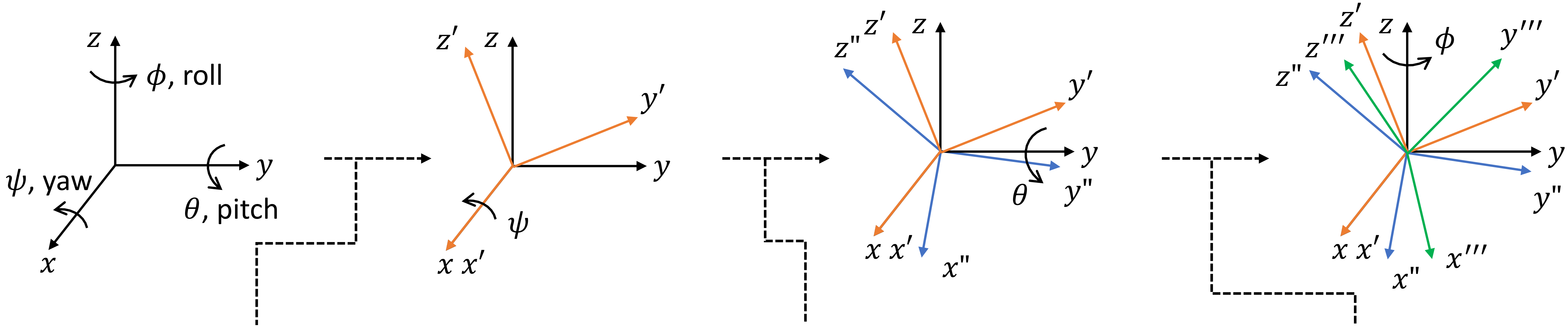
- X-Y-Z fixed angles rotation or Roll-Pitch-Yaw rotation.
- The rotation is made w.r.t. the reference frame.

- Z-Y-X Euler angles rotation.
- The rotation is made w.r.t. the moving frame.

- Z-Y-Z Euler angles rotation.
- The rotation is made w.r.t. the moving frame.

Representation of Orientation

- X-Y-Z fixed angles (Roll-Pitch-Yaw) rotation



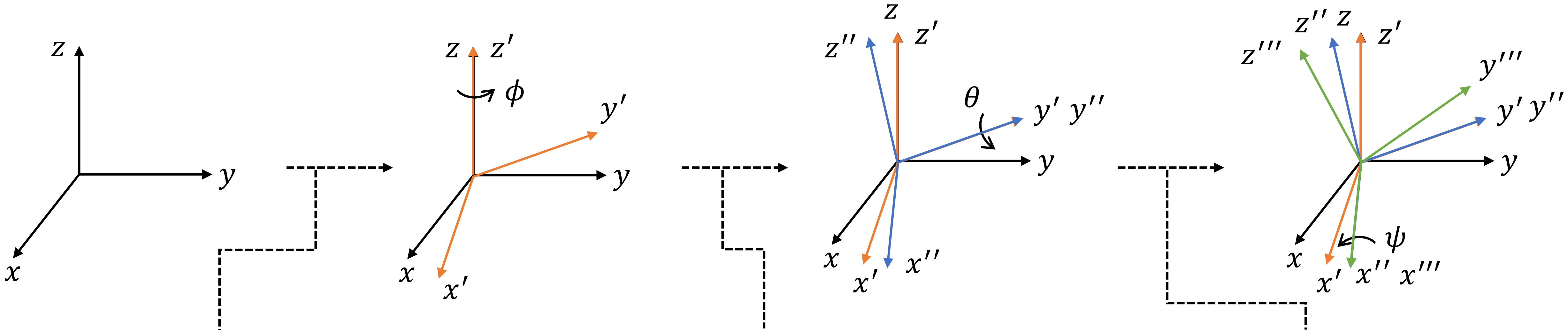
$$\mathrm{Rot}(x,\psi)=\begin{bmatrix}1&0&0&0\\0&\cos(\psi)&-\sin(\psi)&0\\0&\sin(\psi)&\cos(\psi)&0\\0&0&0&1\end{bmatrix}\quad \mathrm{Rot}(y,\theta)=\begin{bmatrix}\cos(\theta)&0&\sin(\theta)&0\\0&1&0&0\\-\sin(\theta)&0&\cos(\theta)&0\\0&0&0&1\end{bmatrix}\quad \mathrm{Rot}(z,\phi)=\begin{bmatrix}\cos(\phi)&-\sin(\phi)&0&0\\\sin(\phi)&\cos(\phi)&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$$

$$\therefore \mathrm{Euler}_{xyz}(\phi,\theta,\psi)\ or\ \mathrm{RPY}(\phi,\theta,\psi)=\mathrm{Rot}(z,\phi)\mathrm{Rot}(y,\theta)\mathrm{Rot}\,(x,\psi)=\begin{bmatrix}n_x&o_x&a_x\\n_y&o_y&a_y\\n_z&o_z&a_z\end{bmatrix}$$

$$\theta=atan2\left(-n_z\,,\sqrt{n_x^2+n_y^2}\right),\phi=atan2\left(\frac{n_y}{\cos(\theta)},\frac{n_x}{\cos(\theta)}\right),\psi=atan2\left(\frac{o_z}{\cos(\theta)},\frac{a_z}{\cos(\theta)}\right)$$

Representation of Orientation

❑ Z-Y-X Euler angles rotation



$$\mathrm{Rot}(z,\phi)=\begin{bmatrix}\cos(\phi) & -\sin(\phi) & 0 & 0\\ \sin(\phi) & \cos(\phi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad \mathrm{Rot}(y,\theta)=\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta) & 0\\ 0 & 1 & 0 & 0\\ -\sin(\theta) & 0 & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad \mathrm{Rot}(x,\psi)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos(\psi) & -\sin(\psi) & 0\\ 0 & \sin(\psi) & \cos(\psi) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\therefore \mathrm{Euler}_{zyx}(\phi,\theta,\psi)=\mathrm{Rot}(z,\phi)\mathrm{Rot}(y,\theta)\mathrm{Rot}\,(x,\psi)=\begin{bmatrix}n_x & o_x & a_x\\ n_y & o_y & a_y\\ n_z & o_z & a_z\end{bmatrix}$$

$$\theta=atan2\left(-n_z\,,\sqrt{n_x^2+n_y^2}\right),\phi=atan2\left(\frac{n_y}{\cos(\theta)},\frac{n_x}{\cos(\theta)}\right),\psi=atan2\left(\frac{o_z}{\cos(\theta)},\frac{a_z}{\cos(\theta)}\right)$$

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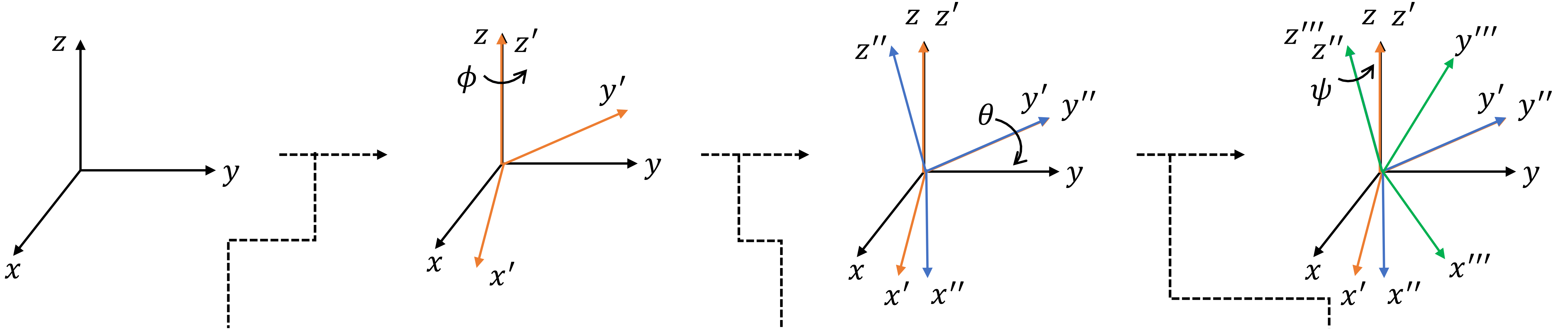
Representation of Orientation

❑ Z-Y-Z Euler angles rotation



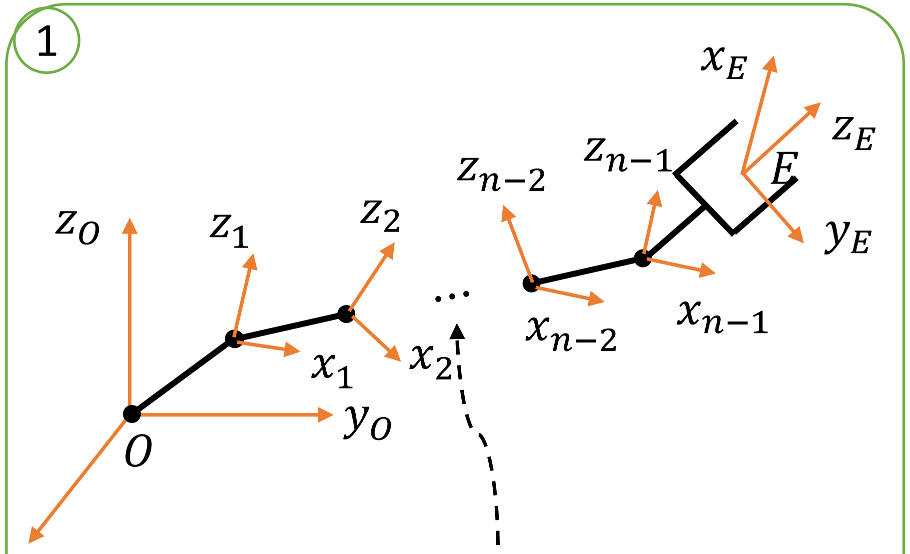
$$Rot(z,\phi)=\begin{bmatrix}\cos(\phi) & -\sin(\phi) & 0 & 0\\ \sin(\phi) & \cos(\phi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad Rot(y,\theta)=\begin{bmatrix}\cos(\theta) & 0 & \sin(\theta) & 0\\ 0 & 1 & 0 & 0\\ -\sin(\theta) & 0 & \cos(\theta) & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\quad Rot(z,\psi)=\begin{bmatrix}\cos(\psi) & -\sin(\psi) & 0 & 0\\ \sin(\psi) & \cos(\psi) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\therefore Euler_{zyz}(\phi,\theta,\psi)=Rot(z,\phi)Rot(y,\theta)Rot\,(z,\psi)=\begin{bmatrix}n_x & o_x & a_x\\ n_y & o_y & a_y\\ n_z & o_z & a_z\end{bmatrix}$$

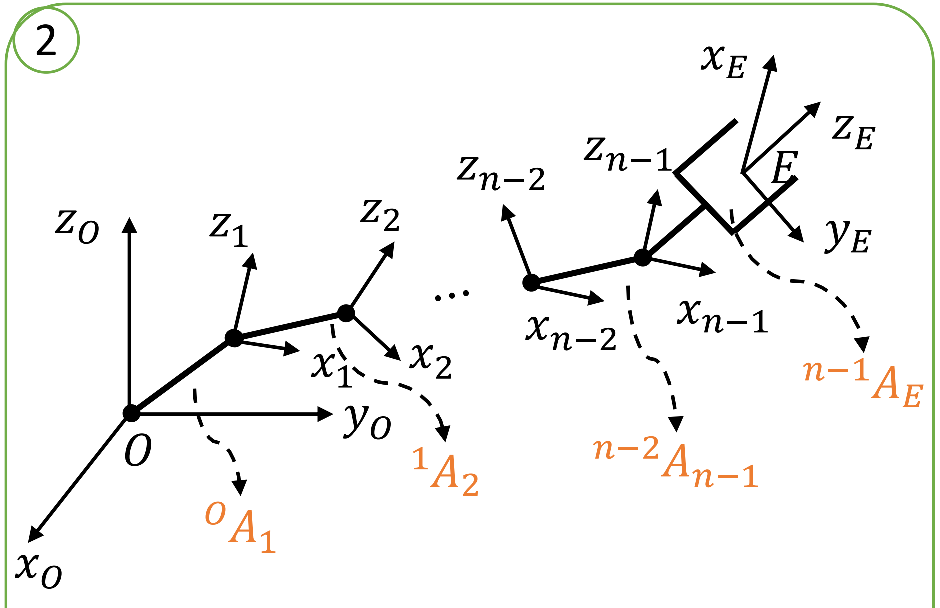
$$\theta=atan2\left(\sqrt{n_z^2+o_z^2}\,,a_z\right),\phi=atan2\left(\frac{a_y}{\sin(\theta)},\frac{a_x}{\sin(\theta)}\right),\psi=atan2\left(\frac{o_z}{\sin(\theta)},-\frac{n_z}{\sin(\theta)}\right)$$

Forward Kinematics

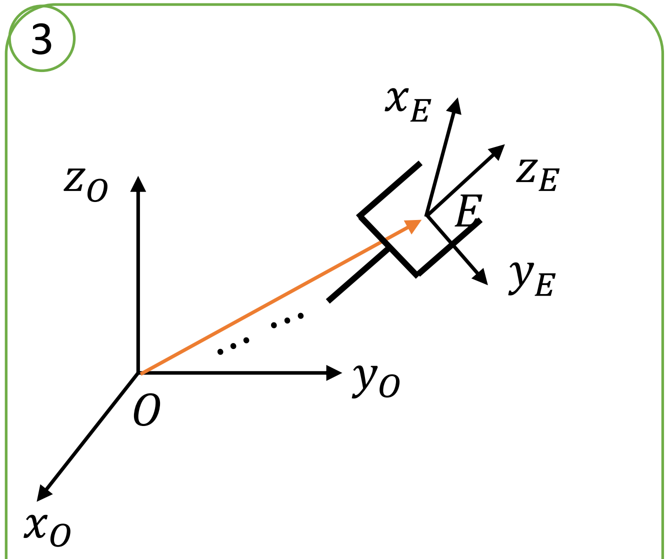
- We will affix **a coordinate frame** to each link of the manipulator. → (1)
- Using homogeneous transformations, we can describe **the relative position and orientation between these frames**.
- **The homogeneous transformation** describing the relation between one link and the next has been called an ***A*** matrix.
- These products of ***A*** matrices have been called ***T*** matrices. → (2)
- Kinematics of robotics is to derive the relationships between **the reference frame** and **the end-effector's frame**. → (3)



An n degree of freedom manipulator



$$T_E = {}^{O}A_1 \, {}^{1}A_2 \, \cdots \, {}^{n-2}A_{n-1} \, {}^{n-1}A_E$$



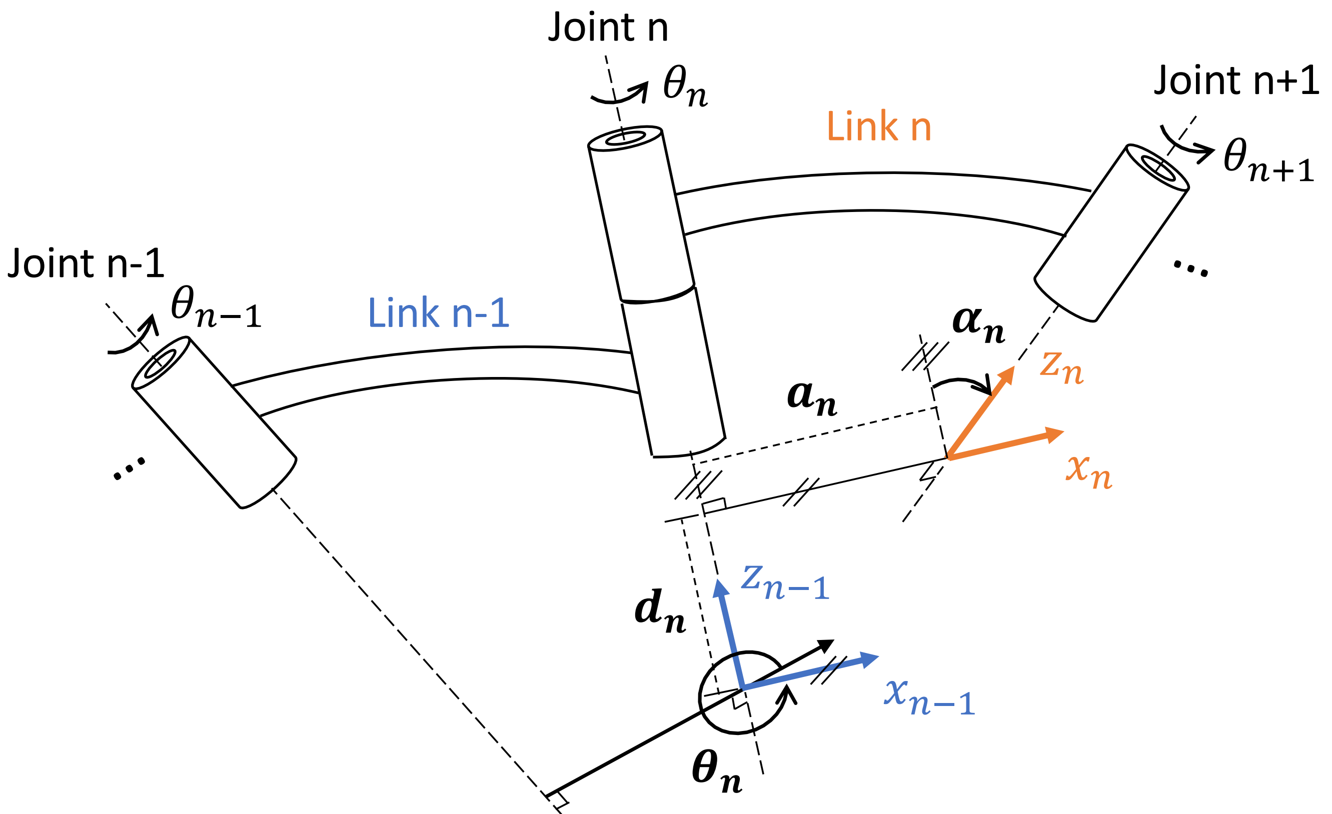
$$T_E = {}^{O}A_1 \, {}^{1}A_2 \, \cdots \, {}^{n-2}A_{n-1} \, {}^{n-1}A_E$$

$$\therefore T_E = {}^{O}A_E$$

Standard Denavit-Hartenberg Notation

❑ How to affix the coordinate frame

- **The origin of the coordinate frame of link n** is set to be at the intersection of **the common normal** between the axes of joints n and n+1 and **the axis of joint n+1**.
- **The *z* axis for link n** will be aligned with **the axis of joint n+1**.
- **The *x* axis for link n** will be directed along the normal **between the axes of joints n and n+1**.



- DH parameters
  - a_n : the distance from z_{n-1} to z_n along x_n.
  - α_n : the angle between z_{n-1} to z_n about x_n.
  - d_n : the distance from x_{n-1} to x_n along z_{n-1}.
  - θ_n : the angle between x_{n-1} to x_n about z_{n-1}.

Standard Denavit-Hartenberg Notation

- How to derive an ***A*** matrix
  - We can establish an ***A*** matrix between successive frame n-1, n by the following rotations and translations.

1. Rotate about z_{n-1}, an angle θ_n
2. Translate along z_{n-1}, a distance d_n
3. Translate along rotated $x_{n-1} = x_n$, a length a_n
4. Rotate about x_n, the twist angle α_n

$$ {}^{n-1}A_n = \underrightarrow{Rot(z,\theta_n)Trans(0,0,d_n)Trans(a_n,0,0)Rot(x,\alpha_n)} $$

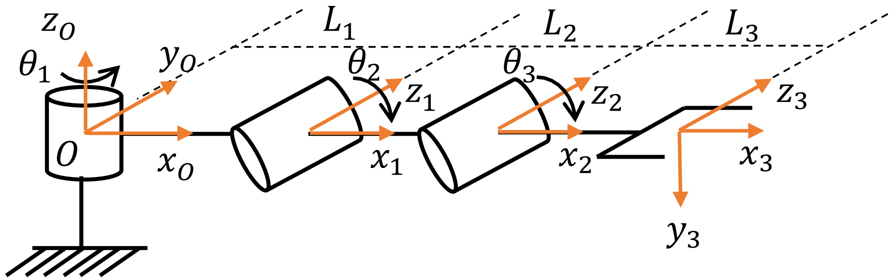
Relative transformation

$$ {}^{n-1}A_n = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) & 0 & 0 \\ \sin(\theta_n) & \cos(\theta_n) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_n \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_n) & -\sin(\alpha_n) & 0 \\ 0 & \sin(\alpha_n) & \cos(\alpha_n) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \therefore {}^{n-1}A_n = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n)\cos(\alpha_n) & \sin(\theta_n)\sin(\alpha_n) & a_n\cos(\theta_n) \\ \sin(\theta_n) & \cos(\theta_n)\cos(\alpha_n) & -\cos(\theta_n)\sin(\alpha_n) & a_n\sin(\theta_n) \\ 0 & \sin(\alpha_n) & \cos(\alpha_n) & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Standard Denavit-Hartenberg Notation

❑ DH parameters table for a 3-DOF manipulator



i	θ_i	α_i	a_i	d_i
1	θ_1	-90°	L_1	0
2	θ_2	0°	L_2	0
3	θ_3	0°	L_3	0

${}^{O}A_1 =$?

${}^{1}A_2 =$?

${}^{2}A_3 =$?

$$\therefore\ {}^{O}T_3 = {}^{O}A_1\ {}^{1}A_2\ {}^{2}A_3$$

Standard Denavit-Hartenberg Notation

- Forward kinematics for a 3-DOF manipulator
  - Set the left-hand side of ${}^{O}T_3 = {}^{O}A_1\ {}^{1}A_2\ {}^{2}A_3$ as the following.

$$ {}^{O}T_3 = \begin{bmatrix} n_x & o_x & a_x & p_x \\ n_y & o_y & a_y & p_y \\ n_z & o_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{R} & \boldsymbol{p} \\ \boldsymbol{0} & 1 \end{bmatrix} $$

  - The result of products of A matrices (i.e. the right-hand side of ${}^{O}T_3 = {}^{O}A_1\ {}^{1}A_2\ {}^{2}A_3$)

$$ \therefore {}^{O}A_1\ {}^{1}A_2\ {}^{2}A_3 = \begin{bmatrix} f_{11}(\boldsymbol{\theta}) & f_{12}(\boldsymbol{\theta}) & f_{13}(\boldsymbol{\theta}) & f_{14}(\boldsymbol{\theta}) \\ f_{21}(\boldsymbol{\theta}) & f_{22}(\boldsymbol{\theta}) & f_{23}(\boldsymbol{\theta}) & f_{24}(\boldsymbol{\theta}) \\ f_{31}(\boldsymbol{\theta}) & f_{32}(\boldsymbol{\theta}) & f_{33}(\boldsymbol{\theta}) & f_{34}(\boldsymbol{\theta}) \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

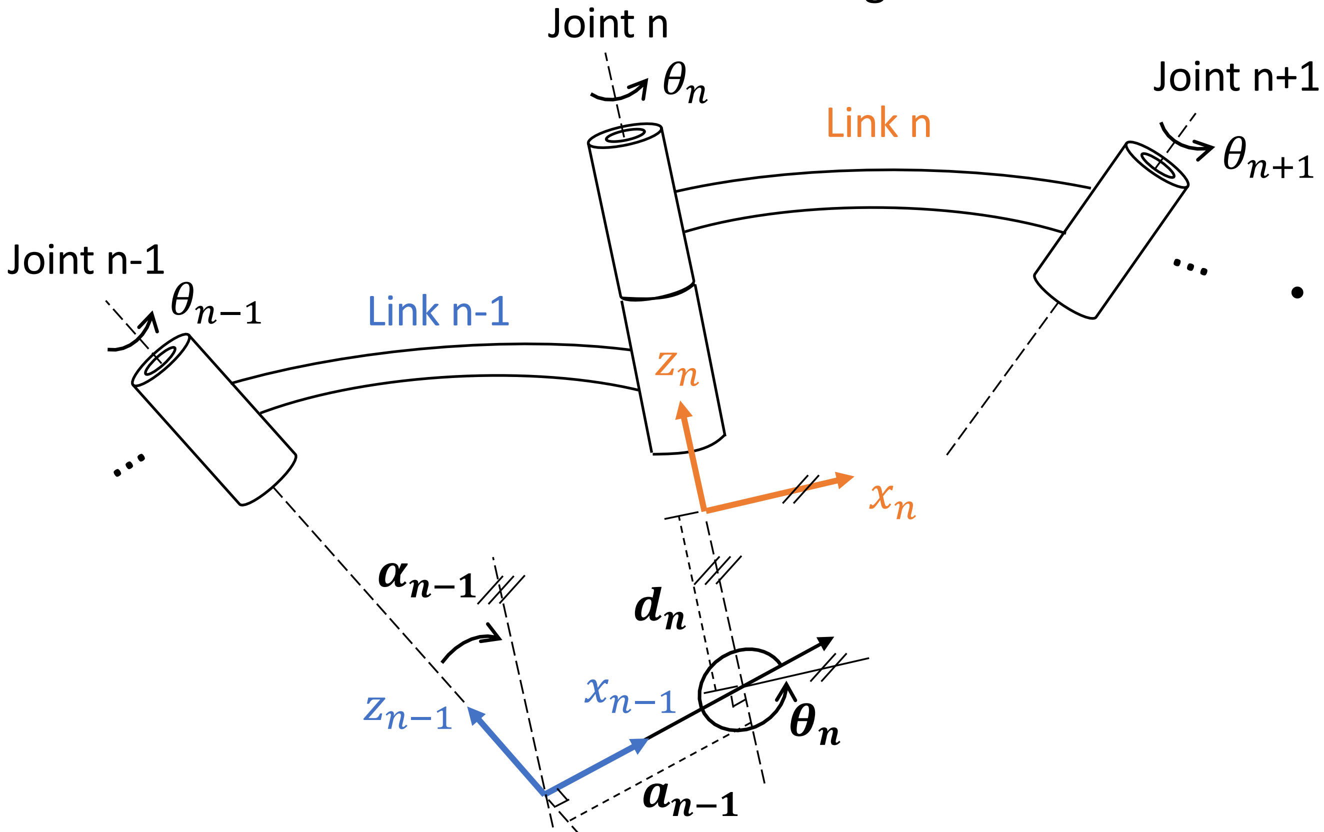
The rotation matrix = $\boldsymbol{R}$ The position vector = $\boldsymbol{p}$

  - Therefore, the position and orientation of the manipulator is represented by the position vector and the rotation matrix.

Modified Denavit-Hartenberg Notation

❑ How to affix the coordinate frame

- **The origin of the coordinate frame of link n** is set to be at the intersection of **the common normal** between the axes of joints n-1 and n and **the axis of joint n**.
- **The *z* axis for link n** will be aligned with **the axis of joint n**.
- **The *x* axis for link n** will be directed along the normal **between the axes of joints n-1 and n**.



- DH parameters
  - a_{n-1} : the distance from z_{n-1} to z_n along x_{n-1}.
  - α_{n-1} : the angle between z_{n-1} to z_n about x_{n-1}.
  - d_n : the distance from x_{n-1} to x_n along z_n.
  - θ_n : the angle between x_{n-1} to x_n about z_n.

Modified Denavit-Hartenberg Notation

❑ How to derive an ***A*** matrix

- We can establish an ***A*** matrix between successive frame n-1, n by the following rotations and translations.

1. Rotate about x_{n-1}, the twist angle α_{n-1}
2. Translate along x_{n-1}, a length a_{n-1}
3. Rotate about z_n, an angle θ_n
4. Translate along z_n, a distance d_n

$$ {}^{n-1}A_n = \underrightarrow{Rot(x, \alpha_{n-1})\, Trans(a_{n-1}, 0,0) Rot(z, \theta_n) Trans(0,0, d_n)} $$

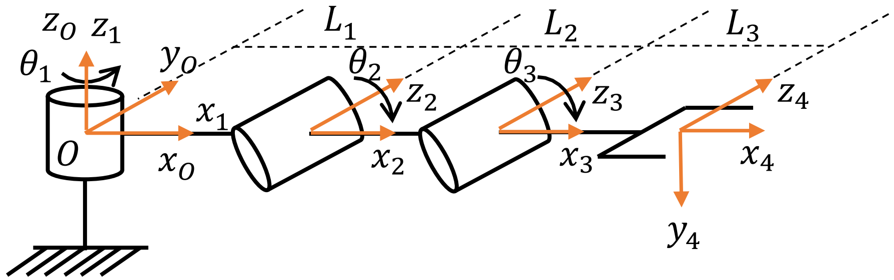
Relative transformation

$$ {}^{n-1}A_n = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\alpha_{n-1}) & -\sin(\alpha_{n-1}) & 0 \\ 0 & \sin(\alpha_{n-1}) & \cos(\alpha_{n-1}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a_{n-1} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) & 0 & 0 \\ \sin(\theta_n) & \cos(\theta_n) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_n \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ \therefore\ {}^{n-1}A_n = \begin{bmatrix} \cos(\theta_n) & -\sin(\theta_n) & 0 & a_{n-1} \\ \sin(\theta_n)\cos(\alpha_{n-1}) & \cos(\theta_n)\cos(\alpha_{n-1}) & -\sin(\alpha_{n-1}) & -d_n\sin(\alpha_{n-1}) \\ \sin(\theta_n)\sin(\alpha_{n-1}) & \cos(\theta_n)\sin(\alpha_{n-1}) & \cos(\alpha_{n-1}) & d_n\cos(\alpha_{n-1}) \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Modified Denavit-Hartenberg Notation

- DH parameters table for a 3-DOF manipulator



i	θ_i	α_{i-1}	a_{i-1}	d_i
1	θ_1	0	0	0
2	θ_2	-90°	L_1	0
3	θ_3	0°	L_2	0
4	0	0°	L_3	0

${}^{O}A_1 =$? ${}^{1}A_2 =$? ${}^{2}A_3 =$? ${}^{2}A_3 =$?

$$\therefore {}^{O}T_4 = {}^{O}A_1\,{}^{1}A_2\,{}^{2}A_3\,{}^{3}A_4$$

Appx. - Denavit-Hartenberg Notation

❑ Video clip to help understand Denavit-Hartenberg (DH) notation

- https://youtu.be/rA9tm0gTln8 : A 'r' parameter in the video is same as an 'a' parameter in this note.