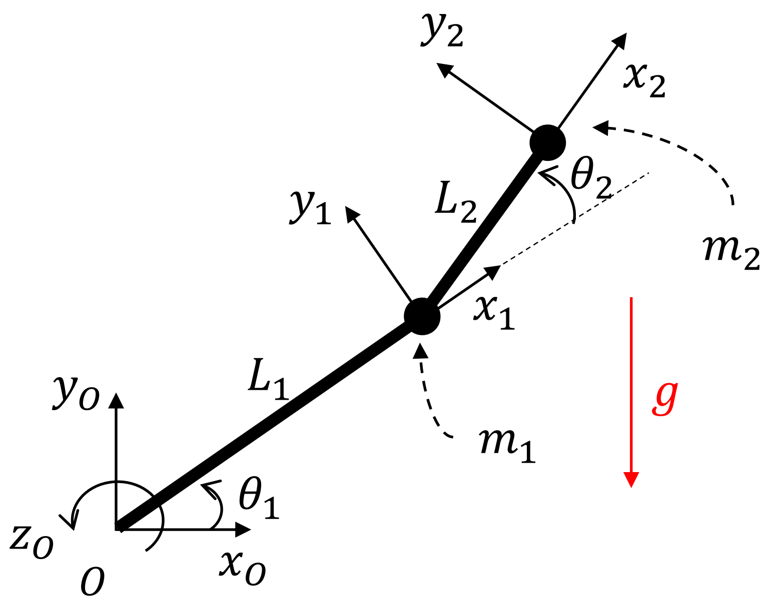
ME729 Advanced Robotics - Homework #6 Solution

3/19/2018

Sangsin Park, Ph.D.

Email me *a pdf file* by next Monday 6 p.m.

1. There is a two-link manipulator. Also the manipulator is affected by GRAVITY.
   Using Lagrangian mechanics, derive equations of motion of the manipulator. [10]



- For link 1, the kinetic and potential energies are

$$K_1 = 1/2\, m_1 L_1^2 \dot{\theta}_1^2$$
$$P_1 = m_1 g L_1 s_1$$

- For link 2, we have

$x_2 = L_1 c_1 + L_2 c_{12}, \quad y_2 = L_1 s_1 + L_2 s_{12}$

$\dot{x}_2 = -L_1 \dot{\theta}_1 s_1 - L_2(\dot{\theta}_1 + \dot{\theta}_2) s_{12}, \quad \dot{y}_2 = L_1 \dot{\theta}_1 c_1 + L_2(\dot{\theta}_1 + \dot{\theta}_2) c_{12}$

so that the velocity squared is

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = L_1^2 \dot{\theta}_1^2 + L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + 2L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) c_2$$

Therefore, the kinetic and potential energies for link 2 is

$$K_2 = 1/2\, m_2 v_2^2 = 1/2\, m_2 L_1^2 \dot{\theta}_1^2 + 1/2\, m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) c_2$$
$$P_2 = m_2 g (L_1 s_1 + L_2 s_{12})$$

- The Lagrangian

$$L = K_1 + K_2 - P_1 - P_2$$
$$= 1/2\,(m_1 + m_2) L_1^2 \dot{\theta}_1^2 + 1/2\, m_2 L_2^2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 L_1 L_2 \left(\dot{\theta}_1^2 + \dot{\theta}_1 \dot{\theta}_2\right) c_2$$
$$-(m_1 + m_2) g L_1 s_1 - m_2 g L_2 s_{12}$$

- Lagrangian's equation

Link 1

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2\dot{\theta}_1 + m_2L_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2L_1L_2(2\dot{\theta}_1 + \dot{\theta}_2)c_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)L_1^2\ddot{\theta}_1 + m_2L_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2L_1L_2(2\ddot{\theta}_1 + \ddot{\theta}_2)c_2 - m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2$$

$$\frac{\partial L}{\partial \theta_1} = -(m_1 + m_2)gL_1c_1 - m_2gL_2c_{12}$$

Link 2

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2L_2^2(\dot{\theta}_1 + \dot{\theta}_2) + m_2L_1L_2\dot{\theta}_1c_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_2} = m_2L_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2L_1L_2\ddot{\theta}_1c_2 - m_2L_1L_2\dot{\theta}_1\dot{\theta}_2s_2$$

$$\frac{\partial L}{\partial \theta_2} = -m_2L_1L_2\left(\dot{\theta}_1^2 + \dot{\theta}_1\dot{\theta}_2\right)s_2 - m_2gL_2c_{12}$$

$$\therefore \tau_1 = \{(m_1 + m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2c_2\}\ddot{\theta}_1 + \{m_2L_2^2 + m_2L_1L_2c_2\}\ddot{\theta}_2 - m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2 + (m_1 + m_2)gL_1c_1 + m_2gL_2c_{12}$$

$$\tau_2 = \{m_2L_2^2 + m_2L_1L_2c_2\}\ddot{\theta}_1 + m_2L_2^2\ddot{\theta}_2 + m_2L_1L_2\dot{\theta}_1^2s_2 + m_2gL_2c_{12}$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1 + m_2)L_1^2 + m_2L_2^2 + 2m_2L_1L_2c_2 & m_2L_2^2 + m_2L_1L_2c_2\\ m_2L_2^2 + m_2L_1L_2c_2 & m_2L_2^2\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}-m_2L_1L_2\left(2\dot{\theta}_1\dot{\theta}_2 + \dot{\theta}_2^2\right)s_2\\ m_2L_1L_2\dot{\theta}_1^2s_2\end{bmatrix} + \begin{bmatrix}(m_1 + m_2)gL_1c_1 + m_2gL_2c_{12}\\ m_2gL_2c_{12}\end{bmatrix}$$